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$\mathcal{H}_\infty$ Control of $\mathcal{LFT}$ Systems: An LMI Approach

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Abstract

The standard $\mathcal{H}_\infty$ control problem for linear state-space systems is extended to general LFT systems, which involve an LFT (Linear Fractional Transformation) on a structured free parameter Δ and can be interpreted as structuredly perturbed uncertain systems. Two generalizations of $\mathcal{H}_\infty$ performance are considered, referred to as μ-performance and $\mathcal{Q}$-performance, with the latter implying the former. Necessary and sufficient conditions for a system to have $\mathcal{Q}$-performance and exist a controller yielding $\mathcal{Q}$-performance can be expressed in terms of structured Linear Matrix Inequalities (LMIs).

1 Introduction

In this paper we are concerned with a class of systems which can be represented as *linear fractional transformations* (LFTs) with respect to some frequency/uncertainty structures. Formally, a system G is an LFT on Δ:

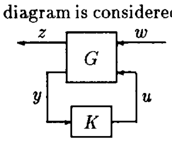
$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \mathcal{F}_u(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right], \Delta) = D + C(\Delta^{-1} - A)^{-1}B.$$

The frequency/uncertainty structure Δ is in a set $\boldsymbol{\Delta} \in \mathbb{C}^{n\times n}$ which has the form

$$\boldsymbol{\Delta} = \{diag[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j\times m_j}\} \subset \mathbb{C}^{n\times n}$$

The various interpretations about Δ were given in Lu et al (1992). Throughout this paper, both plant and controller will be taken as such LFT systems.

The following block diagram is considered in the synthesis problem



where G is the generalized plant with two sets of inputs: the exogenous inputs w and the control inputs u, and two sets of outputs: the measured outputs y and the regulated outputs z. The control problem considered in this paper is to design feedback controller K, which is allowed to have the same dependence on the frequency/uncertainty structure as the *original* plant G, such that the closed loop structure is stable and has a specified performance. In the standard $\mathcal{H}_\infty$ problem, stability means internal stability and the performance is taken to be the $\mathcal{H}_\infty$ norm of the transfer function from w to z. In this paper, we define μ or $\mathcal{Q}$ stability and performance and pursue the controller synthesis problem using these notions. We will focus on $\mathcal{Q}$-case. Packard et al(1991, 1992) also considered $\mathcal{Q}$-performance control problem for a class of LFT systems by transforming the problem into a static feedback $\mathcal{Q}$-stabilization problem, and the solvability conditions are two nicely-coupled LMIs. In this paper, we generalize the approach of Doyle et al (1989) to LFT systems. The resulting $\mathcal{Q}$-performance control problem is solved by considering two LMIs corresponding two simpler problems: full information (FI) and full control (FC), and the controller is constructed by considering FI and FC problems via a separation argument.

This paper is organized as follows: In section 2, we first briefly review some basic properties of LFT systems, especially the LMI characterizations; then $\mathcal{H}_\infty$-performance is generalized to LFT systems using μ and $\mathcal{Q}$ performance, which can be reduced to corresponding augmented stabilization problems. In section 3, the main results are given, two cases are considered, where the static controllers and general controllers are required respectively. In section 4, we will give the constructive proofs to the general case. Other proofs that are straightforward generalizations of existing results are omitted.

2 LFT Systems and LMI Characterization of $\mathcal{H}_\infty$-Performance

Some notions about LFT systems are reviewed here. We refer the reader to Lu et al (1992) and Doyle et al (1991) for more about the properties of LFT Systems, the notions of μ and $\mathcal{Q}$ stability and performance, and the role of LMIs.

A LFT system with a frequency/uncertainty structure $\Delta \in \boldsymbol{\Delta} \subset \mathbb{C}^{n\times n}$ is described as

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \mathcal{F}_u(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right], \Delta) = D + C\Delta(I - A\Delta)^{-1}B.$$

with $(A, B, C, D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{q\times n} \times \mathbf{R}^{q\times p}$, and $\boldsymbol{\Delta}$ can simply be viewed as the usual block diagonal structure set that is standard in the μ theory:

$$\boldsymbol{\Delta} = \{diag[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j\times m_j}\} \subset \mathbb{C}^{n\times n}$$

Note that only complex perturbations are considered in this paper. The results unfortunately do not extend in a nonconservative way to real parameter variations.

The *commutative matrix set* $\mathcal{D}$ of $\boldsymbol{\Delta}$ is defined by

$$\mathcal{D} = \{D \in \mathbb{C}^{n\times n} : D\Delta = \Delta D, \det[D] \neq 0, \Delta \in \boldsymbol{\Delta}\}.$$

A state variable transformation is *admissible* if the transformation matrix $T \in \mathcal{D}$.

Definition 1 (Lu et al, 1992)
(i)The given system is μ-stable *(with respect to* $\boldsymbol{\Delta}$*) if and only if* $\mu_{\boldsymbol{\Delta}}(A) < 1$.
(ii)It is quadratically stable ($\mathcal{Q}$-stable) *(with respect to* $\boldsymbol{\Delta}$*) if and only if* $\mathcal{Q}_{\boldsymbol{\Delta}}(A) := \inf_{D\in\mathcal{D}} \bar{\sigma}(DAD^{-1}) < 1$, *i.e. there is a* $D \in \mathcal{D}$ *such that* $\bar{\sigma}(DAD^{-1}) < 1$.

Note that $\mathcal{Q}$-stability implies μ-stability. While it is often true that the $\mathcal{Q}$ upper bound is a good approximation to μ. Both μ and $\mathcal{Q}$ have interpretations as necessary and sufficient robust stability tests with respect to time-invariant and time-varying perturbations, respectively (this topic is the subject of an invited session in this CDC). There is a large literature on quadratic stability which will not be reviewed in this paper.

It was shown (c.f. Lu et al, 1991 and 1992) that the μ-stability and $\mathcal{Q}$-stability are invariant under admissible state variable transformations. Also that for $\begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$ to be $\mathcal{Q}(\mu)$-stable with respect to structure $\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ it is necessary for A_1 and A_2 to be $\mathcal{Q}(\mu)$-stable with respect to Δ_1 and Δ_2, respectively, and this is sufficient if $A_{12} = 0$ or $A_{21}=0$.

Definition 2 (Lu et al, 1991, 1992)

(i) The given system is $\mathcal{Q}$(or μ)-stabilizable if it can be $\mathcal{Q}$(or μ)-stabilized by "state"-feedback, i.e. there exists a (possibly dynamic) controller $K(\Delta)$ such that $\mathcal{Q}$(or μ)-stabilizes $G_{SF}(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right]$.

(ii) The given system is $\mathcal{Q}$(or μ-detectable) if it can be $\mathcal{Q}$(or μ)-stabilized by output-injection, i.e. there exists a (possibly dynamic) controller $K(\Delta)$ such that $\mathcal{Q}$(or μ)-stabilizes $G_{OI}(\Delta) = \left[\begin{array}{c|c} A & I \\ \hline C & 0 \end{array}\right]$.

Proposition 1 (LMI Characterizations in $\mathcal{Q}$-case)(Lu et al, 1992) *Consider the system $G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with frequency structure Δ, and $(A, B, C, D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{q\times n} \times \mathbf{R}^{q\times p}$.*

(i) It is $\mathcal{Q}$-stable if and only if there exists a $P \in \mathcal{D}$ with $P = P^* > 0$ such that

$$APA^* - P < 0$$

(ii) If $rank(B) = p < n$, assume $B_\perp \in \mathbf{R}^{n\times(n-p)}$ is such that $B^*B_\perp = 0$ and $\begin{bmatrix} B & B_\perp \end{bmatrix}$ is invertible, then the system is $\mathcal{Q}$-stabilizable if and only if there exists a matrix $X \in \mathcal{D}$ with $X = X^* > 0$ such that

$$B_\perp^* AXA^* B_\perp - B_\perp^* X B_\perp < 0.$$

Moreover it can be $\mathcal{Q}$-stabilized by static state-feedback $F = -(B^*X^{-1}B)^{-1}B^*X^{-1}A$.

(iii) If $rank(C) = q < n$, assume $C_\perp \in \mathbf{R}^{(n-q)\times n}$ is such that $C_\perp C^* = 0$ and $\begin{bmatrix} C \\ C_\perp \end{bmatrix}$ is invertible, then the system is $\mathcal{Q}$-detectable if and only if there exists a matrix $Y \in \mathcal{D}$ with $Y = Y^* > 0$ such that

$$C_\perp A^* Y A C_\perp^* - C_\perp Y C_\perp^* < 0.$$

Moreover it can be $\mathcal{Q}$-stabilized by static output-injection $L = -AY^{-1}C^*(CY^{-1}C^*)^{-1}$.

There are two possible generalizations to LFT systems of the standard $\mathcal{H}_\infty$ norm of a transfer function, each corresponding to an extension to robust performance of either μ or $\mathcal{Q}$ stability. For the μ case, denote

$$\|G\|_\infty = \sup_{\Delta\in\mathbf{B\Delta}} \bar{\sigma}(G(\Delta))$$

Let $\mathbf{\Delta}_a = \{\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_c \end{bmatrix} : \Delta \in \mathbf{\Delta}, \Delta_c \in \mathbf{C}^{p\times p}\}$.

If we assume that A is stable, i.e. $\mu_\Delta(A) < 1$, then by main-loop theorem (Doyle et al, 1991), we have that the $\mathcal{H}_\infty$ performance satisfies $\|G\|_\infty < 1$ if and only if $\mu_{\Delta_a}(A_a) < 1$.

In this paper we actually focus on an alternative generalization of $\mathcal{H}_\infty$-performance, called $\mathcal{Q}$-performance. The given system is said to satisfy $\mathcal{Q}$-performance if and only if $\mathcal{Q}_{\Delta_a}(A_a) < 1$.

It can be shown that $\mathcal{Q}$-performance is equivalent to robust performance with time-varying perturbations (Shamma, 1992).

Theorem 1 *(i) The given system G has μ-performance $\|G\|_\infty < 1$ if and only if its augmented system matrix A_a is μ-stable.*

(ii) It satisfies $\mathcal{Q}$-performance if and only if its augmented system matrix A_a is $\mathcal{Q}$-stable.

(iii) $\mathcal{Q}$-performance implies $\|G\|_\infty < 1$.

Remark 1 *That the system satisfies $\mathcal{Q}$-performance is also necessary to the μ-performance to be satisfied if $\mu_{\Delta_a}(A_a) = \mathcal{Q}_{\Delta_a}(A_a)$, but this happens when*

(i) $\mathbf{\Delta} = \{\delta I_n : \delta \in \mathbf{C}\}$, i.e. the one-dimensional systems are considered; or

(ii) $\mathbf{\Delta} = \mathbf{C}^{n\times n}$; or

(iii) $\mathbf{\Delta} = \{diag[\Delta_1, \Delta_2] : \Delta_i \in \mathbf{C}^{n_i\times n_i}\} \subset \mathbf{C}^{n\times n}$.

Theorem 2 (Characterizations of $\mathcal{Q}$-Performance)

Consider the given system and assume $\bar{\sigma}(D) < 1$. Then the following statements are equivalent:

(i) The given system satisfies $\mathcal{Q}$-performance.

(ii) $\mathcal{Q}_{\Delta_a}(A_a) < 1$.

(iii) There exists a positive definite $P \in \mathcal{D}_a$, where $\mathcal{D}_a$ is the commutative matrix set of Δ_a, such that

$$A_a P A_a^* - P < 0$$

(iv) There exists a positive definite $X \in \mathcal{D}$ such that

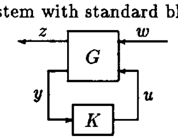
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0$$

(v) There exists a positive definite $X \in \mathcal{D}$ such that $I - D'D - B'XB > 0$ and the following Riccati inequality holds

$$A^*XA - X + (B^*XA + D^*C)^*(I - D^*D - B^*XB)^{-1}(B^*XA + D^*C) + C^*C <$$

3 Solutions to the $\mathcal{Q}$-performance Control Problem

Consider the control system with standard block diagram



Now, assume w, u, z, and y have dimensions p_1, p_2, q_1, and q_2 and assume without loss of generality that $p_1 = q_2$. $G(\Delta)$ with frequency/uncertainty structure $\mathbf{\Delta}$ has a realization (with state x of dimension n) as

$$G(\Delta) = \begin{bmatrix} G_{11}(\Delta) & G_{12}(\Delta) \\ G_{21}(\Delta) & G_{22}(\Delta) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

where all matrices are real and have compatible dimensions with the related physical variables. We further assume $rank(B_2) = p_2 \leq n$ and $rank(C_2) = q_2 \leq n$. In addition, let the state-space realization of $K(\Delta)$ be

$$K(\Delta_0) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

with the frequency/uncertainty structure $\mathbf{\Delta}_0$ which is determined by $\mathbf{\Delta}$. Specially, the controller can have the same dependence on the frequency/uncertainty structure as plant. This can be given a "gain scheduling" interpretation (Packard, 1992), as the controllers depend on the same perturbations as does the plant.

In this paper, we will focus on stability in $\mathcal{Q}$ sense, and (output feedback) controllers K that $\mathcal{Q}$-stabilize systems, i.e. $\mathcal{F}_l(G(\Delta), K(\Delta_0))$ is $\mathcal{Q}$ stable, will be said to be *admissible.* we define its admissible controller set as $\mathcal{K}$, i.e.

$$\mathcal{K} = \{K(\Delta) : \mathcal{F}_l(G(\Delta), K(\Delta_0)) \text{ is } \mathcal{Q}\text{-stable}\}.$$

We also define a subset $\mathcal{K}_s$ of $\mathcal{K}$ as

$$\mathcal{K}_s = \{K \in R^{p_2\times q_2} : \mathcal{F}_l(G(\Delta), K) \text{ is } \mathcal{Q}\text{-stable}\}.$$

The following problem is considered in this paper:

Find a static or dynamical output feedback $K(\Delta) \in \mathcal{K}$ such that the closed-loop system satisfies $\mathcal{Q}$ performance. Note that this implies $\|\mathcal{F}_l(G(\Delta), K(\Delta_0))\|_\infty < 1$.

We will assume that *(A, B_2) is $\mathcal{Q}$-stabilizable and (C_2, A) is $\mathcal{Q}$-detectable*, which is obviously necessary for the solvability for the above problem. We give the solutions in two cases: Static controllers and Dynamic Controllers.

Considered the given systems

$$G(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

$$G_a(\Delta_a) = \left[\begin{array}{cc|c} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right] =: \left[\begin{array}{c|c} A_a & B_a \\ \hline C_a & D_a \end{array}\right]$$

Theorem 3 (Static Controllers)

Consider the given system with $rank(B_2) = p_2 \leq n$, and $rank(C_2) = q_2 \leq n$. Assume that $B_\perp \in \mathbf{R}^{(n+p_1)\times(n+p_1-p_2)}$ is such that $B_a^* B_\perp = 0$ and $\begin{bmatrix} B_a & B_\perp \end{bmatrix}$ is invertible, and $C_\perp \in \mathbf{R}^{(n+p_1-q_2)\times(n+p_1)}$ is such that $C_\perp C_a^* = 0$ and $\begin{bmatrix} C_a \\ C_\perp \end{bmatrix}$ is invertible. There exists an admissible static controller $K \in \mathcal{K}_s$ such that the closed loop system satisfies $\mathcal{Q}$-performance if and only if there exist a positive definite matrices $X \in \mathcal{D}_a$ such that the following two matrix inequalities hold:

$$B_\perp^* A_a X A_a^* B_\perp - B_\perp^* X B_\perp < 0$$

$$C_\perp A_a^* X^{-1} A_a C_\perp^* - C_\perp X^{-1} C_\perp^* < 0.$$

The above theorem was also introduced by Doyle (1984) and Packard et al (1991) in some different forms, and is taken from Lu et al (1992). The controllers can be constructed through the solutions to the two LMIs (see Lu et al, 1992). Since every stabilizing problem with dynamic controllers can be transformed into static controller case (Lu et al, 1991; 1992 and Packard et al, 1991), the solutions can be obtained by statically $\mathcal{Q}$-stabilizing its augment system(Packard et al, 1991 and 1992).

Theorem 4 (Dynamical Controllers)

Consider the given system G with $rank(B_2) = p_2 = p_1 \leq n$, and $rank(C_2) = q_2 \leq n$. Assume that $B_\perp \in \mathbf{R}^{(n+p_1)\times n}$ is such that $B_a^* B_\perp = 0$ and $\begin{bmatrix} B_a & B_\perp \end{bmatrix}$ is invertible in $\mathbf{R}^{(n+p_1)\times(n+p_1)}$, and $C_\perp \in \mathbf{R}^{(n+p_1-q_2)\times(n+p_1)}$ is such that $C_\perp C_a^* = 0$ and $\begin{bmatrix} C_a \\ C_\perp \end{bmatrix}$ is invertible in $\mathbf{R}^{(n+p_1)\times(n+p_1)}$. There exists an admissible controller $K(\Delta_0) \in \mathcal{K}$ such that the closed loop system satisfies the $\mathcal{Q}$-performance if and only if the following two conditions hold,

(i) There exist a positive definite matrix $X = \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} \in \mathcal{D}_a$ such that:

$$B_\perp^* A_a X A_a^* B_\perp - B_\perp^* X B_\perp < 0.$$

Now define $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} = -(B_a^* X^{-1} B_a)^{-1} B_a^* X^{-1} A_a \in \mathbf{R}^{p_1\times(n+p_1)}$ with

$$F_1 = -(B_2^* X_0^{-1} B_2 + D_{12}^* D_{12})^{-1}(B_2^* X_0^{-1} A + D_{12}^* C_1) \in \mathbf{R}^{p_1\times n},$$
$$F_2 = -(B_2^* X_0^{-1} B_2 + D_{12}^* D_{12})^{-1}(B_2^* X_0^{-1} B_1 + D_{12}^* D_{11}) \in \mathbf{R}^{p_1\times p_1},$$

and denote $A_{a_o} = \begin{bmatrix} A & B_1 \\ -F_1 & -F_2 \end{bmatrix}$.

(ii) There a positive definite matrix $Y = \begin{bmatrix} Y_0 & 0 \\ 0 & I \end{bmatrix} \in \mathcal{D}_a$ such that

$$C_\perp A_{a_o}^* Y A_{a_o} C_\perp^* - C_\perp Y C_\perp^* < 0.$$

Now denote $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = -A_{a_o} Y^{-1} C_a^* (C_a Y^{-1} C_a^*)^{-1} \in \mathbf{R}^{(n+p_1)\times p_1}$ with

$$L_1 = -(A Y_0^{-1} C_2^* + B^1 D_{21}^*)(C_2 Y_0^{-1} C_2^* + D_{21} D_{21}^*)^{-1} \in \mathbf{R}^{n\times p_1},$$
$$L_2 = (F_1 Y_0^{-1} C_2^* + F^2 D_{21}^*)(C_2 Y_0^{-1} C_2^* + D_{21} D_{21}^*)^{-1} \in \mathbf{R}^{p_1\times p_1}.$$

When the above two conditions hold, such a controller can be given by

$$K(\Delta) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

with the same frequency structure Δ as the plant, where

$$\hat{A} = A + B_2 F_1 + L_1 C_2 - B_2 L_2 C_2 + (L_1 - B_2 L_2) D_{22} (I + L_2 D_{22})^{-1} (F - L_2 C_2)$$
$$\hat{B} = (L_1 - B_2 L_2)(I + D_{22}(I + L_2 D_{22})^{-1} L_2)$$
$$\hat{C} = (I + L_2 D_{22})^{-1}(-F_1 + L_2 C_2)$$
$$\hat{D} = (I + L_2 D_{22})^{-1} L_2$$

Notice that he controller given in this theorem has a separation structure, and is of "observer form". The last theorem will be proved in the next section.

4 The Construction of $\mathcal{Q}$-Control Problem Solutions

4.1 Performance Control Problems and Stabilization Problems

Consider the system given in the last subsection, we define

$$A_a = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \in \mathbf{R}^{(n+p_1)\times(n+p_1)} \qquad B_a = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \in \mathbf{R}^{(n+p_1)\times p_2}$$

$$C_a = \begin{bmatrix} C_2 & D_{21} \end{bmatrix} \in \mathbf{R}^{q_2\times(n+p_1)} \qquad D_a = D_{22} \in \mathbf{R}^{q_2\times p_2}$$

And define a block structure $\mathbf{\Delta} = \{\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_c \end{bmatrix} : \Delta \in \mathbf{\Delta}, \Delta_c \in \mathbf{C}^{p_1\times p_1}\}$.

From theorem 1, we can conclude that if $K(\Delta_0) \in \mathcal{K}$ makes $\|\mathcal{F}_l(G(\Delta), K(\Delta_0))\|_\infty < 1$ if and only if $K(\Delta_0)$ μ-stabilizes the system

$$G_a(\Delta_a) := \left[\begin{array}{c|c} A_a & B_a \\ \hline C_a & D_a \end{array}\right]$$

Similarly, $K(\Delta_0)$ $\mathcal{Q}$-stabilizes $G_a(\Delta_a)$ if and only if the closed-loop system satisfies $\mathcal{Q}$-performance.

So in this way, the performance control problem is transformed into some stabilization problem. Note that the two stabilization problems are constrained since the controller can only access the frequency structure Δ of G, i.e. partial information of Δ_a of G_a.

4.2 Special Structures

Some special problems are considered here all pertain to the standard block diagram.

Full information (FI)

$$G_{FI}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

Full control (FC)

$$G_{FC}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_1 & D_{11} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right]$$

Disturbance feedforward (DF)

$$G_{DF}(\Delta)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & I & 0 \end{array}\right]$$

Output estimation (OE)

$$G_{OE}(\Delta)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

Note that all of these special systems have the same frequency structures as $G(\Delta)$. We assume all physical variables have the compatible dimensions.

The structures for different problems show clearly that structures FI and FC, as well as DF and OE, are **dual** (Lu et al, 1991). we will also see that structures FI and DF, as well as FC and OE, are equivalent in a sense that will be made precise next.

Proposition 2 *(i)* $K_{FI} := K_{DF}\begin{bmatrix} C_2 & I \end{bmatrix}$ *$\mathcal{Q}$-stabilizes G_{FI} if K_{DF} $\mathcal{Q}$-stabilizes G_{DF}. Furthermore,*

$$\mathcal{F}_\ell(G_{DF},K_{DF})=\mathcal{F}_\ell(G_{FI},K_{DF}\begin{bmatrix} C_2 & I \end{bmatrix}).$$

(ii) Suppose that $A-B_1C_2$ is $\mathcal{Q}$-stable. Then $K_{DF}:=\mathcal{F}_\ell(P_{DF},K_{FI})$ $\mathcal{Q}$-stabilizes G_{DF} if K_{FI} $\mathcal{Q}$-stabilizes G_{FI}. Where

$$P_{DF}(\Delta)=\left[\begin{array}{c|cc} A-B_1C_2 & B_1 & B_2 \\ 0 & 0 & I \\ \hline \begin{bmatrix} I \\ -C_2 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right].$$

Furthermore, $\mathcal{F}_\ell(G_{FI},K_{FI})=\mathcal{F}_\ell(G_{DF},\mathcal{F}_\ell(P_{DF},K_{FI}))$.

Proposition 3 *(i)* $K_{FC} := \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}$ *$\mathcal{Q}$-stabilizes G_{FC} if K_{OE} $\mathcal{Q}$-stabilizes G_{OE}. Furthermore,*

$$\mathcal{F}_\ell(G_{OE},K_{OE})=\mathcal{F}_\ell(G_{FC},\begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}).$$

(ii) Suppose that $A-B_2C_1$ is $\mathcal{Q}$-stable. Then $K_{OE}:=\mathcal{F}_\ell(P_{OE},K_{FC})$ $\mathcal{Q}$-stabilizes G_{OE} if K_{FC} $\mathcal{Q}$-stabilizes G_{FC}. Where

$$P_{OE}(\Delta)=\left[\begin{array}{c|c|c} A-B_2C_1 & 0 & \begin{bmatrix} I & -B_2 \end{bmatrix} \\ \hline C_1 & 0 & \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \\ C_2 & I & \end{array}\right]$$

Furthermore, $\mathcal{F}_\ell(G_{FC},K_{FC})=\mathcal{F}_\ell(G_{OE},\mathcal{F}_\ell(P_{OE},K_{FC}))$.

Note that problems FI and DF (FC and OE) are equivalent in the above sense, since for both structures, we can design the stabilizing controller for any one and the other can be obtained in such a way that the resulting structures internally behave well and their input/output properties are the same.

The $\mathcal{Q}$-control problem for system $G(\Delta)$ will be dealt with as a constrained $\mathcal{Q}$-stabilization problem for the system

$$G_a(\Delta_a)=\left[\begin{array}{cc|c} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right] =: \left[\begin{array}{c|c} A_a & B_a \\ \hline C_a & D_a \end{array}\right].$$

We will use the corresponding special structure in $\mathbf{\Delta}_a$ to each special structure in domain $\mathbf{\Delta}$.

$$G_{FI_a}(\Delta_a)=\left[\begin{array}{cc|c} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ \hline \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] =: \left[\begin{array}{c|c} A_a & B_a \\ \hline I & 0 \end{array}\right];$$

$$G_{FC_a}(\Delta_a)=\left[\begin{array}{cc|c} A & B_1 & \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ C_1 & D_{11} & \\ \hline C_2 & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right] =: \left[\begin{array}{c|c} A_a & I \\ \hline C_a & 0 \end{array}\right];$$

$$G_{DF_a}(\Delta_a)=\left[\begin{array}{cc|c} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ \hline C_2 & I & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_a & B_a \\ \hline C_{a_o} & 0 \end{array}\right];$$

$$G_{OE_a}(\Delta_a)=\left[\begin{array}{cc|c} A & B_1 & B_2 \\ C_1 & D_{11} & I \\ \hline C_2 & D_{21} & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_a & B_{a_o} \\ \hline C_a & 0 \end{array}\right].$$

Note that G_{FI_a} and G_{FC_a} are actually state-feedback and output-injection structures for G_a in frequency structure domain $\mathbf{\Delta}_a$.

4.3 A Construction of $\mathcal{Q}$-Controllers

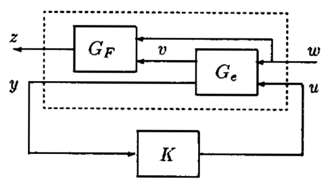
Without loss of generality, we further assume the realization of $G(\Delta)$ has the structure that $D_{22}=0$.

Consider the system $G_a(\Delta_a)$, by the assumption (i), we know that $G_a(\Delta_a)$ is $\mathcal{Q}$-stabilizable, and a solution to LMI in (i) provide a constant "state feedback" $F=\begin{bmatrix} F_1 & F_2 \end{bmatrix}$ such that $A_a+B_aF=\begin{bmatrix} A+B_2F_1 & B_1+B_2F_2 \\ C_1+D_{12}F_1 & D_{11}+D12F_2 \end{bmatrix}$ are $\mathcal{Q}$-stable with respect to $\mathbf{\Delta}_a$. I.e. $\left[\begin{array}{c|c} A+B_2F_1 & B_1+B_2F_2 \\ \hline C_1+D_{12}F_1 & D_{11}+D_{12}F_2 \end{array}\right]$ satisfies $\mathcal{Q}$-performance.

Now we consider the original system $G(\Delta)$, denote x its state. Note that $F=\begin{bmatrix} F_1 & F_2 \end{bmatrix}$ is actually a special FI stabilizing controller. Now let

$$v=u-F_1x-F_2w$$

Then the system can be broken into two subsystems $G_F(\Delta)$ and $G_e(\Delta)$ as shown pictorially below



with

$$G_F(\Delta)=\left[\begin{array}{c|cc} A+B_2F_1 & B_1+B_2F_2 & B_2 \\ \hline C_1+D_{12}F_1 & D_{11}+D_{12}F_2 & D_{12} \end{array}\right]$$

which is $\mathcal{Q}$-stable and the $\mathcal{Q}$-performance of transfer matrix from w to z is less than 1, and

$$G_e(\Delta)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F_1 & -F_2 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

Note that the latter system G_e is of OE structure. Now we claim that:

Lemma 1 *(i)K $\mathcal{Q}$(or μ)-stabilizes G and makes $\|\mathcal{F}_l(G(\Delta),K(\Delta))\|_\infty<1$ if and only if K $\mathcal{Q}$(or μ)-stabilizes G_e and $\|\mathcal{F}_l(G_e(\Delta),K(\Delta))\|_\infty<1$.*

(ii)K $\mathcal{Q}$(or μ)-stabilizes G and makes the corresponding closed-loop system satisfies $\mathcal{Q}$-performance if and only if K $\mathcal{Q}$(or μ)-stabilizes G_e and makes the corresponding closed-loop system satisfies $\mathcal{Q}$-performance.

Proof. Since the stabilities can be verified by the fact that the cascade system is $\mathcal{Q}(\mu)$-stable if and only if their composed subsystems are (remark 1). So we only need to verify the $\mathcal{H}_\infty$-performance.

Now we play this in domain $\mathbf{\Delta}_a$. The corresponding system $G_{F_a}(\Delta_a)$ in domain $\mathbf{\Delta}_a$ to $G_F(\Delta)$ is a system without output,

$$G_{F_a}(\Delta_a) = \left[\begin{array}{cc|c} A + B_2F_1 & B_1 + B_2F_2 & B_2 \\ C_1 + D_{12}F_1 & D_{11} + D_{12}F_2 & D_{12} \end{array}\right]$$

which has input from some output v of system $G_{e_a}(\Delta_a)$ which is the corresponding system in domain $\mathbf{\Delta}_a$ to $G_e(\Delta)$,

$$G_{e_a}(\Delta) = \left[\begin{array}{cc|c} A & B_1 & B_2 \\ -F_1 & -F_2 & I \\ \hline C_2 & D_{21} & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_{a_o} & B_{a_o} \\ \hline C_a & 0 \end{array}\right]$$

Note that the above cascade connection in domain $\mathbf{\Delta}_a$ is exactly the structure $G_a(\Delta_a)$. So assume $K(\Delta)$ is chosen, then the closed loop system in domain $\mathbf{\Delta}_a$ is $\mathcal{S}(G_a(\Delta_a), K(\Delta))$. Since according to the above argument $\mathcal{S}(G_a(\Delta_a), K(\Delta))$ is a cascade connection between $G_{F_a}(\Delta_a)$ and $\mathcal{S}(G_{e_a}(\Delta_a), K(\Delta))$. Now by the choice of $F = \left[\begin{array}{cc} F_1 & F_2 \end{array}\right]$ we know that $G_{F_a}(\Delta_a)$ is $\mathcal{Q}$-stable, so it is μ-stable. So still by remark 1, if $K(\Delta)$ is chosen such that $\|\mathcal{F}_l(G(\Delta), K(\Delta))\|_\infty < 1$ (or the corresponding closed-loop system satisfies $\mathcal{Q}$-performance), or $\mathcal{S}(G_a(\Delta_a), K(\Delta))$ is μ(or $\mathcal{Q}$)-stable, if and only if $\mathcal{S}(G_{e_a}(\Delta_a), K(\Delta))$ is μ(or $\mathcal{Q}$)-stable, i.e. $\|\mathcal{F}_l(G_e(\Delta), K(\Delta))\|_\infty < 1$ (or the corresponding closed-loop system satisfies $\mathcal{Q}$-performance).

$\square$

So we transformed the OF problem to a OE problem.

Now we consider G_e which is of OE structure in domain $\mathbf{\Delta}$.

$$G_e(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F_1 & -F_2 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

Since $A + B_2F_1$ is $\mathcal{Q}$-stable with respect to Δ by remark 1. So by proposition 3, we know the above problem is equivalent to a *FC* problem,

$$G_{FC_e}(\Delta) = \left[\begin{array}{c|c|c} A & B_1 & \left[\begin{array}{cc} I & 0 \end{array}\right] \\ \hline -F_1 & -F_2 & \left[\begin{array}{cc} 0 & I \end{array}\right] \\ C_2 & D_{21} & \left[\begin{array}{cc} 0 & 0 \end{array}\right] \end{array}\right]$$

whose corresponding system in domain $\mathbf{\Delta}_a$ is $\left[\begin{array}{c|c} A_{a_o} & I \\ \hline C_a & 0 \end{array}\right]$. The $\mathcal{Q}$-stabilization with respect to Δ_a for the latter system, which is of output-injection structure solves the $\mathcal{H}_\infty$-control problem for the *FC* problem. The condition (ii) in the main theorem say that this problem is solvable, and the static output-inject controller $L = \left[\begin{array}{c} L_1 \\ L_2 \end{array}\right]$ can be obtained by solving the LMI in (ii) as given in the theorem statement. So a controller for $\hat{K}(\Delta)$the *OE* problem is given by

$$\hat{K}(\Delta) = \mathcal{F}_\ell(J(\Delta), \left[\begin{array}{c} L_1 \\ L_2 \end{array}\right])$$

where

$$J(\Delta) = \left[\begin{array}{c|c|c} A + B_2F_1 & 0 & \left[\begin{array}{cc} I & -B_2 \end{array}\right] \\ \hline -F_1 & 0 & \left[\begin{array}{cc} 0 & I \end{array}\right] \\ C_2 & I & \left[\begin{array}{cc} 0 & 0 \end{array}\right] \end{array}\right]$$

This produces

$$\hat{K}(\Delta) = \left[\begin{array}{c|c} A + B_2F_1 + L_1C_2 - B_2L_2C_2 & L_1 - B_2L_2 \\ \hline -F_1 + L_2C_2 & L_2 \end{array}\right]$$

Note that by proposition 3, we have $\mathcal{F}_\ell(G_e(\Delta), \hat{K}(\Delta)) = \mathcal{F}_\ell(G_{FC_e}(\Delta), L)$, so both closed loop-systems satisfy the $\mathcal{Q}$-performances and the external properties $\|\mathcal{F}_\ell(G_e(\Delta), \hat{K}(\Delta))\|_\infty = \|\mathcal{F}_\ell(G_{FC_e}(\Delta), L)\|_\infty < 1$.

Now we drop the assumption $D_{22} = 0$ and we get the given controller for the general problem.

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